

Unification of Non-Abelian $SU(N)$ Gauge Theory and Gravitational Gauge Theory

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Abstract

In this paper, a general theory on unification of non-Abelian $SU(N)$ gauge interactions and gravitational interactions is discussed. $SU(N)$ gauge interactions and gravitational interactions are formulated on the similar basis and are unified in a semi-direct product group $GSU(N)$. Based on this model, we can discuss unification of fundamental interactions of Nature.

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1 Introduction

It is known that there are four kinds of fundamental interactions in Nature, which are strong interactions, electromagnetic interactions, weak interactions and gravitational interactions. All these fundamental interactions can be described by gauge field theories, which can be regarded as the common nature of all these fundamental interactions. It provides us the possibility to unify different kinds of fundamental interactions in the framework of gauge theory. The first unification of fundamental interactions in human history is the unification of electric interactions and magnetic interactions, which is made by Maxwell in 1864. Now, we know that electromagnetic theory is a $U(1)$ Abelian gauge theory. In 1921, H.Weyl tried to unify electromagnetic interactions and gravitational interactions in a theory which has local scale invariance[1, 2]. Weyl's original work is not successful, however in his great attempt, he introduced one of the most important concept in modern physics: gauge transformation and gauge symmetry. After the foundation of quantum mechanics, V.Fock, H.Weyl and W.Pauli found that quantum electrodynamics is a $U(1)$ gauge invariant theory[3, 4, 5].

In 1954, Yang and Mills proposed non-Abelian gauge field theory[6]. Soon after, non-Abelian gauge field theory is applied to elementary particle theory. In about 1967 and 1968, using the spontaneously symmetry breaking and Higgs mechanism[7, 8, 9, 10, 11, 12, 13, 14], S.Weinberg and A.Salam proposed the unified electroweak theory, which is a $SU(2) \times U(1)$ gauge theory[15, 16, 17]. The predictions of unified electroweak theory have been confirmed in a large number of experiments, and the intermediate gauge bosons W^\pm and Z^0 which are predicted by unified electroweak model are also found in experiments. From nineteen seventies, physicist begin studying Grand Unified theories which try to unify strong, electromagnetic and weak interactions in a simple group. At that time, $SU(5)$ model[18, 19], $SO(10)$ model[20, 21, 22], E_6 model[23, 24, 25] and other models[26, 27, 28] are proposed. In these attempts, gravitational interactions are not considered.

Gauge treatment of gravity was suggested immediately after the gauge theory birth itself[29, 30, 31]. In the traditional gauge treatment of gravity, Lorentz group is localized, and the gravitational field is not represented by gauge potential[32, 33, 34]. It is represented by metric field. The theory has beautiful mathematical forms, but up to now, its renormalizability is not proved. In other words, it is conventionally considered to be perturbatively non-renormalizable. Recently, some new attempts were proposed to use Yang-Mills theory to reformulate quantum gravity[35, 36, 37, 38, 39]. In these new approaches, the importance of gauge fields is emphasized. Some physicists also try to use gauge potential to represent gravitational field, some suggest that we should pay more attention on translation group.

Recently, Wu proposed a new quantum gauge theory of gravity which is a renormalizable quantum gravity[40]. Based on gauge principle, space-time translation group is selected to be the gravitational gauge group. After localization of gravitational gauge group, the gravitational field appears as the corresponding gauge potential. In this paper, we will discuss unification of fundamental interactions which is based on gravitational gauge theory. In literature [42], Wu proposed a model in which $U(1)$ gauge theory is consistently unified with gravitational gauge theory in a semi-direct product group. In this paper, we will generalize the unification from $U(1)$ gauge group to $SU(N)$ gauge group, i.e., the unification of general non-Abelian $SU(N)$ gauge interactions and gravitational gauge interactions is studied. If $N = 3$, this theory is just the unified theory of strong and gravitational interactions.

2 Lagrangian

The generators of $SU(N)$ group is denoted as T_a , they satisfies

$$[T_a, T_b] = if_{abc}T_c, \quad (2.1)$$

$$Tr(T_a T_b) = K\delta_{ab}. \quad (2.2)$$

The $SU(N)$ non-Abelian gauge field is denoted as A_μ , which is an element of $SU(N)$ Lie algebra,

$$A_\mu(x) = A_\mu^a(x)T_a, \quad (2.3)$$

where $A_\mu^a(x)$ are component fields.

Because an arbitrary element $U(x)$ of $SU(N)$ group does not commute with an arbitrary element \hat{U}_ϵ of gravitational gauge group,

$$[U(x), \hat{U}_\epsilon] \neq 0. \quad (2.4)$$

The product group of $SU(N)$ group and gravitational group is not direct product group, but semi-direct product group, which we will denoted as $GSU(N)$

$$GSU(N) \triangleq SU(N) \otimes_s \text{Gravitational Gauge Group}. \quad (2.5)$$

An arbitrary element of $GSU(N)$ is denoted as $g(x)$, which is defined by

$$g(x) \triangleq \hat{U}_\epsilon \cdot U(x). \quad (2.6)$$

The gauge covariant derivative of $GSU(N)$ group is

$$\mathbf{D}_\mu \triangleq \partial_\mu - igC_\mu - ig_s A_\mu = D_\mu - ig_s A_\mu, \quad (2.7)$$

where g_s is the coupling constant of non-Abelian $SU(N)$ gauge interactions, C_μ is the gravitational gauge field and D_μ is the gravitational gauge covariant derivative

$$D_\mu = \partial_\mu - igC_\mu(x). \quad (2.8)$$

Gravitational gauge field which is a vector in gauge group space,

$$C_\mu(x) = C_\mu^\alpha(x)\hat{P}_\alpha, \quad (2.9)$$

where $\hat{P}_\alpha = -i\frac{\partial}{\partial x^\alpha}$ is the generator of gravitational gauge group.

The field strength of non-Abelian gauge field A_μ is

$$A_{\mu\nu} = (D_\mu A_\nu) - (D_\nu A_\mu) - ig_s[A_\mu, A_\nu]. \quad (2.10)$$

$A_{\mu\nu}$ is also an element of $SU(N)$ Lie algebra,

$$A_{\mu\nu}(x) = A_{\mu\nu}^a(x)T_a, \quad (2.11)$$

where

$$A_{\mu\nu}^a = (D_\mu A_\nu^a) - (D_\nu A_\mu^a) + g_s f_{abc} A_\mu^b A_\nu^c. \quad (2.12)$$

$A_{\mu\nu}$ is not a $SU(N)$ gauge covariant field strength. In order to define $SU(N)$ gauge covariant field strength, we need a matrix G which is given by

$$G = (G_\mu^\alpha) = (\delta_\mu^\alpha - gC_\mu^\alpha) = I - gC. \quad (2.13)$$

Its inverse matrix is denoted as G^{-1} ,

$$G^{-1} = \frac{1}{I - gC} = (G_\alpha^{-1\mu}). \quad (2.14)$$

They satisfy the following relations,

$$G_\mu^\alpha G_\alpha^{-1\nu} = \delta_\mu^\nu, \quad (2.15)$$

$$G_\beta^{-1\mu} G_\mu^\alpha = \delta_\beta^\alpha. \quad (2.16)$$

It can be proved that

$$D_\mu = G_\mu^\alpha \partial_\alpha. \quad (2.17)$$

The field strength of gravitational gauge field is defined by

$$F_{\mu\nu} \triangleq \frac{1}{-ig}[D_\mu, D_\nu]. \quad (2.18)$$

Its explicit expression is

$$F_{\mu\nu}(x) = \partial_\mu C_\nu(x) - \partial_\nu C_\mu(x) - igC_\mu(x)C_\nu(x) + igC_\nu(x)C_\mu(x). \quad (2.19)$$

$F_{\mu\nu}$ is also a vector in gauge group space,

$$F_{\mu\nu}(x) = F_{\mu\nu}^\alpha(x) \cdot \hat{P}_\alpha, \quad (2.20)$$

where

$$F_{\mu\nu}^\alpha = \partial_\mu C_\nu^\alpha - \partial_\nu C_\mu^\alpha - gC_\mu^\beta(\partial_\beta C_\nu^\alpha) + gC_\nu^\beta(\partial_\beta C_\mu^\alpha). \quad (2.21)$$

$SU(N)$ Gauge covariant field strength is defined by

$$\mathbf{A}_{\mu\nu} = A_{\mu\nu} + gG_\sigma^{-1\lambda}A_\lambda F_{\mu\nu}^\sigma = \mathbf{A}_{\mu\nu}^a T_a, \quad (2.22)$$

where

$$\mathbf{A}_{\mu\nu}^a = A_{\mu\nu}^a + gG_\sigma^{-1\lambda}A_\lambda^a F_{\mu\nu}^\sigma. \quad (2.23)$$

The lagrangian density \mathcal{L}_0 is given by,

$$\begin{aligned} \mathcal{L}_0 = & -\bar{\psi}[\gamma^\mu(D_\mu - ig_s A_\mu) + m]\psi - \frac{1}{4}\eta^{\mu\rho}\eta^{\nu\sigma}\mathbf{A}_{\mu\nu}^a\mathbf{A}_{\rho\sigma}^a \\ & - \frac{1}{4}\eta^{\mu\rho}\eta^{\nu\sigma}g_{\alpha\beta}F_{\mu\nu}^\alpha F_{\rho\sigma}^\beta, \end{aligned} \quad (2.24)$$

where

$$g_{\alpha\beta} \triangleq \eta_{\mu\nu}(G^{-1})_\alpha^\mu(G^{-1})_\beta^\nu. \quad (2.25)$$

The full lagrangian density \mathcal{L} is defined by

$$\mathcal{L} = J(C)\mathcal{L}_0, \quad (2.26)$$

where

$$J(C) = \sqrt{-\det g_{\alpha\beta}} \quad (2.27)$$

is a special factor to resume gravitational gauge symmetry of the system. The action is

$$S = \int d^4x \mathcal{L} = \int d^4x J(C)\mathcal{L}_0. \quad (2.28)$$

3 Gauge Symmetry

Now, we discuss symmetry of the system. Under $SU(N)$ gauge transformations, gravitational gauge field $C_\mu(x)$ is kept unchanged. Therefore, $F_{\mu\nu}^\alpha$, G_μ^α , $G_\alpha^{-1\mu}$, D_μ and $J(C)$ are not changed under local $SU(N)$ gauge transformations. Other fields and operators transform as

$$\psi \rightarrow \psi' = (U(x)\psi), \quad (3.1)$$

$$A_\mu \rightarrow A'_\mu = U(x)A_\mu U^{-1}(x) - \frac{1}{ig_s}U(x)(D_\mu U^{-1}(x)), \quad (3.2)$$

$$A_{\mu\nu} \rightarrow A'_{\mu\nu} = U(x)A_{\mu\nu}U^{-1}(x) + \frac{g}{ig_s}F_{\mu\nu}^\sigma U(x)(\partial_\sigma U^{-1}(x)), \quad (3.3)$$

$$\mathbf{A}_{\mu\nu} \rightarrow \mathbf{A}'_{\mu\nu} = U(x)\mathbf{A}_{\mu\nu}U^{-1}(x). \quad (3.4)$$

Using all these relations, we can prove that the lagrangian density \mathcal{L}_0 does not change under local $SU(N)$ gauge transformations

$$\mathcal{L}_0 \rightarrow \mathcal{L}'_0 = \mathcal{L}_0. \quad (3.5)$$

Because both integration measure d^4x and $J(C)$ are not changed under non-Abelian $SU(N)$ gauge transformation, the action is invariant under $SU(N)$ gauge transformation. Therefore, the system has local $SU(N)$ gauge symmetry.

Under local gravitational gauge transformation, the transformations of various fields and operators are

$$\psi \rightarrow \psi' = (\hat{U}_\epsilon \psi), \quad (3.6)$$

$$\bar{\psi} \rightarrow \bar{\psi}' = (\hat{U}_\epsilon \bar{\psi}), \quad (3.7)$$

$$A_\mu \rightarrow A'_\mu = \hat{U}_\epsilon A_\mu \hat{U}_\epsilon^{-1}, \quad (3.8)$$

$$C_\mu \rightarrow C'_\mu = \hat{U}_\epsilon C_\mu \hat{U}_\epsilon^{-1} - \frac{1}{ig} \hat{U}_\epsilon (\partial_\mu \hat{U}_\epsilon^{-1}), \quad (3.9)$$

$$g_{\alpha\beta} \rightarrow g'_{\alpha\beta} = \Lambda_\alpha^{\alpha_1} \Lambda_\beta^{\beta_1} (\hat{U}_\epsilon g_{\alpha_1\beta_1}), \quad (3.10)$$

$$\mathbf{D}_\mu \rightarrow \mathbf{D}'_\mu = \hat{U}_\epsilon \mathbf{D}_\mu \hat{U}_\epsilon^{-1}, \quad (3.11)$$

$$A_{\mu\nu} \rightarrow A'_{\mu\nu} = \hat{U}_\epsilon A_{\mu\nu} \hat{U}_\epsilon^{-1}, \quad (3.12)$$

$$F_{\mu\nu}^\sigma \rightarrow F'_{\mu\nu}{}^\sigma = \Lambda_\rho^\sigma \hat{U}_\epsilon F_{\mu\nu}^\rho \hat{U}_\epsilon^{-1}, \quad (3.13)$$

$$G_\mu^\alpha \rightarrow G'_\mu{}^\alpha = \Lambda_\beta^\alpha \hat{U}_\epsilon G_\mu^\beta \hat{U}_\epsilon^{-1}, \quad (3.14)$$

$$G_\alpha^{-1\mu} \rightarrow G'^{-1\mu}_\alpha = \Lambda_\alpha^\beta \hat{U}_\epsilon G_\beta^{-1\mu} \hat{U}_\epsilon^{-1}, \quad (3.15)$$

$$\mathbf{A}_{\mu\nu} \rightarrow \mathbf{A}'_{\mu\nu} = \hat{U}_\epsilon \mathbf{A}_{\mu\nu} \hat{U}_\epsilon^{-1}, \quad (3.16)$$

$$J(C) \rightarrow J'(C') = J \cdot \hat{U}_\epsilon J(C) \hat{U}_\epsilon^{-1}, \quad (3.17)$$

where J is the Jacobian of the corresponding transformation which is given by

$$J = \det \left(\frac{\partial(x - \epsilon)^\mu}{\partial x^\nu} \right), \quad (3.18)$$

and Λ^α_β and Λ_α^β are the transformation matrices which are given by [40, 41]:

$$\Lambda^\alpha_\beta = \frac{\partial x^\alpha}{\partial(x - \epsilon(x))^\beta}, \quad (3.19)$$

$$\Lambda_\alpha^\beta = \frac{\partial(x - \epsilon(x))^\beta}{\partial x^\alpha}. \quad (3.20)$$

Using all these relations and the following relation

$$\int d^4x J(\hat{U}_\epsilon f(x)) = \int d^4x f(x), \quad (3.21)$$

where $f(x)$ is an arbitrary function, we can prove that

$$\mathcal{L}_0 \rightarrow \mathcal{L}'_0 = (\hat{U}_\epsilon \mathcal{L}_0), \quad (3.22)$$

$$\mathcal{L} \rightarrow \mathcal{L}' = J(\hat{U}_\epsilon \mathcal{L}), \quad (3.23)$$

$$S \rightarrow S' = S. \quad (3.24)$$

Therefore, the system has local gravitational gauge symmetry.

Combining above results on local $SU(N)$ gauge transformations and local gravitational gauge transformations, we know that under general $GSU(N)$ gauge transformation $g(x)$, transformations of various fields and operators are

$$\psi \rightarrow \psi' = (g(x)\psi), \quad (3.25)$$

$$\bar{\psi} \rightarrow \bar{\psi}' = (\hat{U}_\epsilon(\bar{\psi}U^\dagger(x))), \quad (3.26)$$

$$A_\mu \rightarrow A'_\mu = g(x) \left[A_\mu - \frac{1}{ig_s} (D_\mu U^{-1}(x)) U(x) \right] g^{-1}(x), \quad (3.27)$$

$$C_\mu \rightarrow C'_\mu = \hat{U}_\epsilon C_\mu \hat{U}_\epsilon^{-1} - \frac{1}{ig} \hat{U}_\epsilon (\partial_\mu \hat{U}_\epsilon^{-1}), \quad (3.28)$$

$$\mathbf{D}_\mu \rightarrow \mathbf{D}'_\mu = g(x) \mathbf{D}_\mu g^{-1}(x), \quad (3.29)$$

$$A_{\mu\nu} \rightarrow A'_{\mu\nu} = g(x) \left[A_{\mu\nu} + \frac{g}{ig_s} F_{\mu\nu}^\sigma (\partial_\sigma U^{-1}(x)) U(x) \right] g^{-1}(x), \quad (3.30)$$

$$F_{\mu\nu}^\sigma \rightarrow F_{\mu\nu}'^\sigma = \Lambda^\sigma_\rho g(x) F_{\mu\nu}^\rho g^{-1}(x), \quad (3.31)$$

$$G_\mu^\alpha \rightarrow G_\mu'^\alpha = \Lambda^\alpha_\beta g(x) G_\mu^\beta g^{-1}(x), \quad (3.32)$$

$$G_\alpha^{-1\mu} \rightarrow G_\alpha'^{-1\mu} = \Lambda_\alpha^\beta g(x) G_\beta^{-1\mu} g^{-1}(x), \quad (3.33)$$

$$g_{\alpha\beta} \rightarrow g'_{\alpha\beta} = \Lambda_\alpha^{\alpha_1} \Lambda_\beta^{\beta_1} \cdot g(x) g_{\alpha_1\beta_1} g^{-1}(x), \quad (3.34)$$

$$\mathbf{A}_{\mu\nu} \rightarrow \mathbf{A}'_{\mu\nu} = g(x) \mathbf{A}_{\mu\nu} g^{-1}(x), \quad (3.35)$$

$$J(C) \rightarrow J'(C') = J \cdot g(x) J(C) g^{-1}(x). \quad (3.36)$$

Action S is invariant local $GSU(N)$ gauge transformation.

4 Interactions

The lagrangian density \mathcal{L} can also be separated into two parts: the free lagrangian density \mathcal{L}_F and interaction lagrangian density \mathcal{L}_I

$$\mathcal{L} = \mathcal{L}_F + \mathcal{L}_I, \quad (4.1)$$

where

$$\begin{aligned} \mathcal{L}_F = & -\bar{\psi}(\gamma^\mu \partial_\mu + m)\psi - \frac{1}{4}\eta^{\mu\rho}\eta^{\nu\sigma}\eta_{\alpha\beta}F_{0\mu\nu}^\alpha F_{0\rho\sigma}^\beta \\ & - \frac{1}{4}\eta^{\mu\rho}\eta^{\nu\sigma}A_{0\mu\nu}^a A_{0\rho\sigma}^a, \end{aligned} \quad (4.2)$$

$$\begin{aligned}
\mathcal{L}_I = & -(J(C) - 1)\bar{\psi}(\gamma^\mu \partial_\mu + m)\psi \\
& -\frac{1}{4}\eta^{\mu\rho}\eta^{\nu\sigma}(J(C)g_{\alpha\beta} - \eta_{\alpha\beta})F_{0\mu\nu}^\alpha F_{0\rho\sigma}^\beta \\
& -\frac{1}{4}\eta^{\mu\rho}\eta^{\nu\sigma}(J(C) - 1)A_{0\mu\nu}^a A_{0\rho\sigma}^a \\
& +gJ(C)\bar{\psi}\gamma^\mu(\partial_\alpha\psi)C_\mu^\alpha + ig_s J(C)\bar{\psi}\gamma^\mu T_a\psi A_\mu^a \\
& +g\eta^{\mu\rho}\eta^{\nu\sigma}J(C)(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)C_\rho^\alpha(\partial_\alpha A_\sigma^a) \\
& -\frac{g_s}{2}\eta^{\mu\rho}\eta^{\nu\sigma}J(C)f_{abc}A_\rho^b A_\sigma^c(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) \\
& +g\eta^{\mu\rho}\eta^{\nu\sigma}g_{\alpha\beta}J(C)(\partial_\mu C_\nu^\alpha - \partial_\nu C_\mu^\alpha)C_\rho^\gamma(\partial_\gamma C_\sigma^\beta) \\
& -\frac{g}{2}\eta^{\mu\rho}\eta^{\nu\sigma}J(C)G_\alpha^{-1\lambda}A_\lambda^a A_{\mu\nu}^a F_{\rho\sigma}^\alpha \\
& -\frac{g_s^2}{4}\eta^{\mu\rho}\eta^{\nu\sigma}J(C)f_{abc}f_{ab_1c_1}A_\mu^b A_\nu^c A_\rho^{b_1} A_\sigma^{c_1} \\
& +gg_s\eta^{\mu\rho}\eta^{\nu\sigma}J(C)f_{abc}A_\mu^b A_\nu^c C_\rho^\alpha(\partial_\alpha A_\sigma^a) \\
& -\frac{g^2}{2}\eta^{\mu\rho}\eta^{\nu\sigma}J(C)(C_\mu^\alpha\partial_\alpha A_\nu^a - C_\nu^\alpha\partial_\alpha A_\mu^a)C_\rho^\beta(\partial_\beta A_\sigma^a) \\
& -\frac{g^2}{2}\eta^{\mu\rho}\eta^{\nu\sigma}g_{\alpha\beta}J(C)(C_\mu^\gamma\partial_\gamma C_\nu^\alpha - C_\nu^\gamma\partial_\gamma C_\mu^\alpha)C_\rho^\delta(\partial_\delta C_\sigma^\beta) \\
& -\frac{g^2}{4}\eta^{\mu\rho}\eta^{\nu\sigma}J(C)G_\alpha^{-1\lambda}G_\beta^{-1\kappa}A_\lambda^a A_\kappa^a F_{\mu\nu}^\alpha F_{\rho\sigma}^\beta.
\end{aligned} \tag{4.3}$$

In above relations, $A_{0\mu\nu}^a$ and $F_{0\mu\nu}^\alpha$ are defined by

$$A_{0\mu\nu}^a = (\partial_\mu A_\nu^a) - (\partial_\nu A_\mu^a). \tag{4.4}$$

$$F_{0\mu\nu}^\alpha = (\partial_\mu C_\nu^\alpha) - (\partial_\nu C_\mu^\alpha). \tag{4.5}$$

The explicit expression for $J(C)$ is

$$J(C) = 1 + \sum_{m=1}^{\infty} \frac{1}{m!} \left(\sum_{n=1}^{\infty} \frac{g^n}{n} \text{tr}(C^m) \right)^m \tag{4.6}$$

From eq.(4.2), we can write out propagators of Dirac field, $SU(N)$ non-Abelian gauge field and gravitational gauge field. From eq.(4.3), we can write out Feynman rules for various interaction vertexes and calculate Feynman diagrams for various interaction precesses. We can also see that, because of the influence of the factor $J(C)$, matter fields can directly couple to arbitrary number of gravitational gauge field, which is important for the renormalization of the theory.

5 Equations of Motion and Energy-Momentum Tensor

The equation of motion of Dirac field is

$$(\gamma^\mu \mathbf{D}_\mu + m)\psi = 0. \quad (5.1)$$

The equation of motion of $SU(N)$ gauge field is

$$\partial^\mu \mathbf{A}_{\mu\nu}^a = -g_s \eta_{\nu\sigma} J_a^\sigma, \quad (5.2)$$

where

$$\begin{aligned} J_a^\nu &= i\bar{\psi}\gamma^\nu T_a \psi + \eta^{\nu\rho}\eta^{\nu_1\sigma} f_{abc} A_{\nu_1}^c \mathbf{A}_{\rho\sigma}^b - \frac{g}{g_s} \eta^{\mu_1\rho} \eta^{\nu\sigma} \partial_\mu (C_{\mu_1}^\mu \mathbf{A}_{\rho\sigma}^a) \\ &\quad - \frac{g}{2g_s} \eta^{\mu_1\rho} \eta^{\nu_1\sigma} G_{\sigma_1}^{-1\nu} F_{\mu_1\nu_1}^{\sigma_1} \mathbf{A}_{\rho\sigma}^a + \frac{g}{g_s} \eta^{\mu\rho} \eta^{\nu\sigma} G_\tau^{-1\lambda} (D_\mu C_\lambda^\tau) \mathbf{A}_{\rho\sigma}^a. \end{aligned} \quad (5.3)$$

J_a^ν is a conserved current,

$$\partial_\nu J_a^\nu = 0. \quad (5.4)$$

When gravitational gauge field vanishes, the above current J_a^ν returns to the conventional current in traditional non-Abelian $SU(N)$ gauge field theory, which is

$$J_a^\nu = i\bar{\psi}\gamma^\nu T_a \psi + \eta^{\nu\rho} \eta^{\mu\sigma} f_{abc} A_\mu^c A_{\rho\sigma}^b. \quad (5.5)$$

But if gravitational gauge field does not vanish, because of the influence from gravitational gauge field, the conventional current eq.(5.5) is no longer a conserved current.

The equation of motion of gravitational gauge field is

$$\partial^\mu (\eta^{\nu\sigma} g_{\alpha\beta} F_{\mu\sigma}^\beta) = -g T_{g\alpha}^\nu, \quad (5.6)$$

where $T_{g\beta}^\sigma$ is the gravitational energy-momentum tensor

$$\begin{aligned} T_{g\alpha}^\nu &= \bar{\psi}\gamma^\nu \partial_\alpha \psi - \eta^{\mu_1\rho} \eta^{\nu\sigma} g_{\alpha_1\beta} F_{\rho\sigma}^\beta (\partial_\alpha C_{\mu_1}^{\alpha_1}) - \eta^{\mu_1\rho} \eta^{\nu\sigma} \mathbf{A}_{\rho\sigma}^a (\partial_\alpha A_{\mu_1}^a) \\ &\quad + G_\alpha^{-1\nu} \mathcal{L}_0 - \frac{1}{2} \eta^{\mu\rho} \eta^{\lambda\sigma} g_{\alpha\beta} G_\gamma^{-1\nu} F_{\mu\lambda}^\beta F_{\rho\sigma}^\gamma \\ &\quad - \eta^{\mu_1\rho} \eta^{\nu\sigma} \partial_\mu (g_{\alpha\beta} C_{\mu_1}^\mu F_{\rho\sigma}^\beta) + \eta^{\mu_1\rho} \eta^{\nu\sigma} \partial_\mu (G_\alpha^{-1\lambda} G_{\mu_1}^\mu A_\lambda^a \mathbf{A}_{\rho\sigma}^a) \\ &\quad + \eta^{\mu\rho} \eta^{\nu\sigma} g_{\alpha\beta} G_\tau^{-1\lambda} (D_\mu C_\lambda^\tau) F_{\rho\sigma}^\beta + g \eta^{\mu\rho} \eta^{\nu\sigma} G_\tau^{-1\lambda_1} (D_\mu C_{\lambda_1}^\tau) G_\alpha^{-1\lambda} A_\lambda^a \mathbf{A}_{\rho\sigma}^a \\ &\quad - \frac{g}{2} \eta^{\mu_1\rho} \eta^{\nu_1\sigma} G_{\sigma_1}^{-1\nu} G_\alpha^{-1\lambda} A_\lambda^a \mathbf{A}_{\rho\sigma}^a F_{\mu_1\nu_1}^{\sigma_1} - g \eta^{\mu_1\rho} \eta^{\nu\sigma} G_{\sigma_1}^{-1\lambda} A_\lambda^a (\partial_\alpha C_{\mu_1}^{\sigma_1}) \mathbf{A}_{\rho\sigma}^a. \end{aligned} \quad (5.7)$$

The global gravitational gauge symmetry of the system gives out another energy-momentum tensor which is called inertial energy-momentum tensor,

$$T_{i\alpha}^\mu = J(C) \left(\bar{\psi} \gamma^\nu G_\nu^\mu (\partial_\alpha \psi) + \eta^{\mu 1 \rho} \eta^{\nu \sigma} G_{\mu 1}^\mu \mathbf{A}_{\rho\sigma}^a (\partial_\alpha A_\nu^a) + \eta^{\mu \rho} \eta^{\nu \sigma} g_{\beta\gamma} F_{\rho\sigma}^\gamma (\partial_\alpha C_\nu^\beta) + \delta_\alpha^\mu \mathcal{L}_0 \right. \\ \left. - g \eta^{\mu 1 \rho} \eta^{\nu \sigma} g_{\beta\gamma} C_{\mu 1}^\mu F_{\rho\sigma}^\gamma (\partial_\alpha C_\nu^\beta) + g \eta^{\mu 1 \rho} \eta^{\nu \sigma} G_\beta^{-1\lambda} G_{\mu 1}^\mu A_\lambda^a \mathbf{A}_{\rho\sigma}^a (\partial_\alpha C_\nu^\beta) \right). \quad (5.8)$$

Compare eq.(5.7) with eq.(5.8), we can see that the inertial energy-momentum tensor is not equivalent to the gravitational energy-momentum tensor. In this case, they are not equivalent even when gravitational field vanishes. When gravitational field vanishes, the gravitational energy-momentum tensor becomes $T_{0g\alpha}^\nu$,

$$T_{0g\alpha}^\nu = \bar{\psi} \gamma^\nu \partial_\alpha \psi - \eta^{\mu\rho} \eta^{\nu\sigma} A_{\rho\sigma}^a (\partial_\alpha A_\mu^a) + \eta_{1\alpha}^\nu \mathcal{L}_0 + \eta^{\nu\sigma} \partial^\mu (A_\alpha^a A_{\mu\sigma}^a), \quad (5.9)$$

while inertial energy-momentum tensor becomes $T_{0i\alpha}^\nu$

$$T_{0i\alpha}^\nu = \bar{\psi} \gamma^\nu \partial_\alpha \psi - \eta^{\mu\rho} \eta^{\nu\sigma} A_{\rho\sigma}^a (\partial_\alpha A_\mu^a) + \delta_\alpha^\nu \mathcal{L}_0. \quad (5.10)$$

Therefore, we have

$$T_{0g\alpha}^\nu = T_{0i\alpha}^\nu + \eta^{\nu\sigma} \partial^\mu (A_\alpha^a A_{\mu\sigma}^a). \quad (5.11)$$

But this difference has no contribution on energy-momentum. The spatial integration of time component of energy-momentum tensor gives out energy-momentum of the system. The inertial energy-momentum $P_{0i\alpha}$ is

$$P_{0i\alpha} = \int d^3x T_{0i\alpha}^0, \quad (5.12)$$

and the gravitational energy-momentum $P_{0g\alpha}$ is

$$P_{0g\alpha} = \int d^3x T_{0g\alpha}^0. \quad (5.13)$$

So, their difference is

$$P_{0g\alpha} - P_{0i\alpha} = - \int d^3x \partial_i (A_\alpha^a A_{i0}^a) = 0. \quad (5.14)$$

It means that, when gravitational gauge field vanishes, equivalence principle holds.

6 Summary

In this paper, we have studied unifications of ordinary $SU(N)$ gauge interactions with gravitational gauge interactions, which is unified in the semi-direct product

group $GSU(N)$. Because generators of ordinary $SU(N)$ group and generators of gravitational gauge group have different dimensions, that is, generators of $SU(N)$ group are dimensionless while generators of gravitational gauge group have length dimension, it is hard to unify $SU(N)$ gauge interactions and gravitational gauge interactions in a simple group. Because of the difference of dimensions of generators, we need at least two independent parameters for coupling constant in any kind of unified theory. Because when we unify $SU(N)$ gauge interactions and gravitational gauge interactions in $GSU(N)$ group, we only need two independent parameters for coupling constant, this unified theory can be regarded as a minimal theory of unification. It is impossible to unify four kinds of fundamental interactions in a simple group in which only one independent coupling constant is used.

Because $SU(N)$ gauge group and gravitational gauge group are unified in a semi-direct product group, not in a direct product group, field strength of gravitational gauge field joins into the definition of gauge covariant field strength of $SU(N)$ gauge field. This will cause additional interactions between $SU(N)$ gauge fields and gravitational gauge field, which even cause that gravitational energy-momentum tensor is not equivalent to inertial energy-momentum tensor when gravitational field vanishes, but this difference does not affect the equivalence of gravitational mass and inertial mass when gravitational field vanishes.

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